

Selected Article

Study of the Transient Motion of Capillarity in Constant Section Tubes

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CAD-CAE team of Information Technology Dept. in Products Design & Engineering Center is engaged in CAE calculations for products designs as one of the main works. We have put importance in them to find out from CAE results the technical information and the directions in which designs should take for high quality products. IN order to carry out them and be always ready to reply requests from wider engineering fields than before, it is necessary for CAE team to understand basic mechanical engineering knowledge more deeply and continue to make an effort to widen our skills for CAE. Through such our team thought as describe above, study of the transient motion of capillarity in constant section tubes is introduced in this paper as one example in which the technical information were got by the theoretical analysis before using CAE and afterword the results were compared with ones got by CAE calculations.

Introduction

Some analyzers collect their samples to be analyzed by capillary tubes using capillarity. Generally it is favorable for the sampling time to be short and for the sampling volume to be small as possible. In order to carry out them figuring out the basic characteristic of the capillarity, theoretical study of the transient motion of capillarity without gravity effect was held, and the results from them were compared with ones by CAE calculation. And it was also tried to indicate the direction to satisfy the design requirements above.

Analysis Model and Parameters

The image used for this theoretical analysis is shown in Figure 1. At first a tube which has constant rectangular section in the flow direction was studied.

Parameters used are as below.

- h : Height of the liquid surface [m]
- a : Length of the rectangular section longer side [m]
- b : Length of the rectangular section shorter side [m]
- D : Diameter of the circular section of tubes [m]
- ρ : Density of the liquid [m^3/kg]
- σ : Surface tension of the liquid [N/m]

θ : Contact angle of the liquid [$^\circ$]

μ : Viscosity of the liquid [Pa·s]

t : Time [s]

Though the surface of the liquid has a curvature actually, it is omitted in Figure 1 for simplicity.

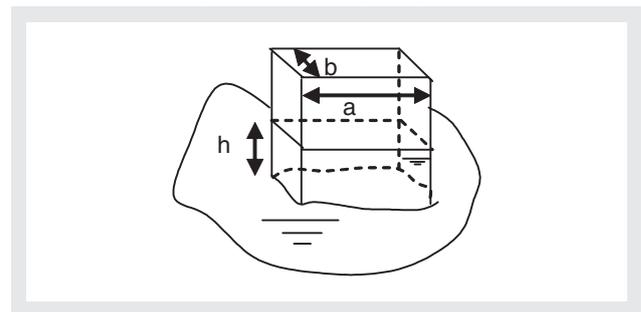


Figure 1 Analysis model

In Case without Gravity and Friction on the Wall

Equation of Motion and its Solution

Now in order to understand the basic transient motion of capillarity, an analysis without friction on the wall and gravity that obstruct the liquid motion (in case for

capillary tubes being set standing opposite to gravity) was done.

One end of the tube touched the liquid in a big reservoir and the momentum of the liquid in the reservoir can be neglected because of its slow motion.

Suppose the surface of the liquid is sucked at the height of h at the time of $t=t$, the average velocity in the arbitrary height section of the liquid in the tube is constant by continuity equation for the tubes with constant section configuration in the flow direction. This constant average velocity should be as below.

The average velocity = \dot{h} (1)

Here, $\dot{}$ indicates time rate of change.

So the total momentum of the liquid in the tube is shown as below.

The total momentum of the liquid = $abh\rho\dot{h}$ (2)

The force by the surface tension which generates the liquid motion is given as

The force for motion = $2(a+b)\sigma\cos\theta$ (3)

From Equation (2) (3) ,we obtain below by Newton's second law.

$$\frac{d}{dt}(abh\rho\dot{h}) = 2(a+b)\sigma\cos\theta \quad \dots\dots\dots (4)$$

Now letting A as below,

$$A = \frac{2(a+b)\sigma\cos\theta}{ab\rho} \quad \dots\dots\dots (5)$$

A is time-constant, so Equation (4) can be changed into as below.

$$\frac{d^2}{dt^2}\left(\frac{1}{2}h^2\right) = A \quad \dots\dots\dots (6)$$

By integrating Equation (6) twice with consideration of h is positive number , we can obtains Equation (7).

$$h = \sqrt{A(t+c_2)^2 + c_1} \quad \dots\dots\dots (7)$$

c_1, c_2 are integration constants

As a conclusion the velocity of the liquid surface is given as below.

$$\dot{h} = \frac{A(t+c_2)}{h} \quad \dots\dots\dots (8)$$

Characteristic of the Transient Motion without Friction and Gravity

Now, letting $t \rightarrow \infty$ in Equation (7), h becomes as below.

$$h \rightarrow \sqrt{A}(t+c_2) \quad \dots\dots\dots (9)$$

The right-hand side of Equation (9) increases linearly with time , which means the velocity is time-constant or the surface speed approach constant number of \sqrt{A} as time passes.

In order to prevent the velocity from being infinite at the time of $t=0$ in Equation (8), we assume that the height of L [m] of liquid exists with the velocity of 0 at the time of $t=0$, so integration constants become $c_1=L^2, c_2=0$. Figure 2 shows Equation (7) (8) with the initial conditions of $L=3$ mm and the surface velocity=0 at the time $t=0$ for instance. Triangle marks indicate height of the surface and circle marks indicate the surface velocity. The liquid is water and each parameters are as below.

$a=1e-3, b=2e-4, \rho=998, \sigma=7.28e-2, \theta=0$

Then \sqrt{A} becomes as blow.

$$\sqrt{A} = 0.9353 \quad \dots\dots\dots (10)$$

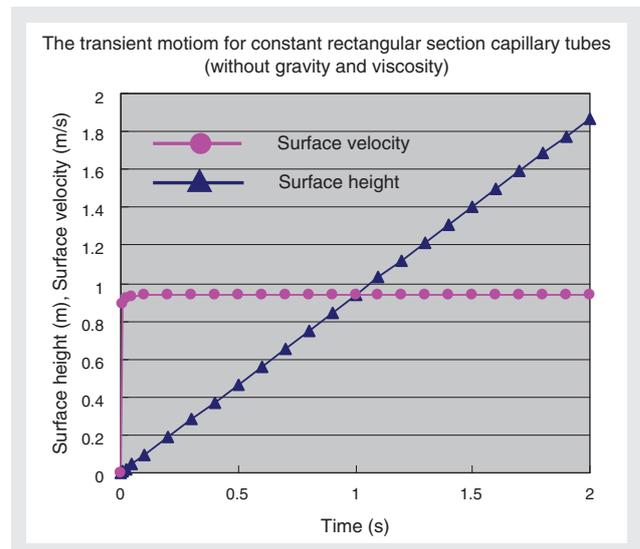


Figure 2 Surface transient actions without gravity and friction

Though L was tried to be changed from 3 mm to 0.01 mm in Figure 2 as a reference, L only affects the gradient of the surface velocity during a very short time just after $t=0$ and brings almost no influence especially to the transient motion of the surface height h . So the velocity can generally be thought to reach \sqrt{A} instantly and afterwards

the surface be thought to go up with the constant speed of \sqrt{A} . The discussion above can be applied to the circular tube and \sqrt{A} in this case is shown as Equation (11).

$$\sqrt{A} = \sqrt{\frac{4\sigma\cos\theta}{D\rho}} \dots\dots\dots (11)$$

As a consequence above, the basic transient motion of capillarity without friction and gravity is thought to have the constant surface velocity during almost all time. Also both friction and gravity affect the motion only as obstruction (in case for tubes being set standing opposite to gravity), so \sqrt{A} can be seemed to be an ideal velocity, or a maximum velocity except just after $t=0$. In conclusion a shorter sampling time than one caused with the velocity of \sqrt{A} turns out to be very hard in the sampling devices applying the capillary tubes with constant section configuration.

In Case with Friction on the Wall

Equation of Motion and its Solution

In the next step friction by viscosity of the liquid is going to be considered also. Generally the representative length for flow is small and the velocity of the liquid is not so big in capillary tubes, so the flow state can be regarded as laminar flow. So friction force can be thought to be proportional to the flow rate as seen in laminar flow-meters. Also considering that the liquid area touching the wall increases as the height of the surface increases, the friction acting on the liquid in the tube increases in proportional to the height of the surface. In conclusion the friction force can be thought to be linear with both the surface velocity and the surface height of the liquid at every time. Now we consider rectangular section with supposition $a \gg b$, the friction force by viscosity of the liquid can be the same as one of the flow between the infinitely large parallel planes. So applying the theory of incompressible fluid laminar flow, the friction force acting on the fluid whose height is h is as below.

The friction force = $\frac{12a\mu}{b} h\dot{h} \dots\dots\dots (12)$

In conclusion the corresponding equation of motion to Equation (4) is given as below.

$$\frac{d}{dt}(abh\dot{h}) = -\frac{12a\mu}{b} h\dot{h} + 2(a+b)\sigma\cos\theta \dots\dots\dots (13)$$

Now letting

$$B = \frac{12\mu}{b^2\rho} \dots\dots\dots (14)$$

B becomes time-constant., and Equation (13) can be change into as below using A in Equation (5).

$$\frac{d}{dt}(h\dot{h}) = -Bh\dot{h} + A \dots\dots\dots (15)$$

also letting

$$h\dot{h} = X \dots\dots\dots (16)$$

equation of motion also can be written as below.

$$\frac{dX}{dt} = -BX + A \dots\dots\dots (17)$$

This is an ordinary differential equations for X, and X can be solved analytically, or h can be analytically. Considering h is positive the solution is shown as below.

$$h = \sqrt{\frac{2}{B^2} \exp(-Bt+c_3) + \frac{2A}{B} t + c_4} \dots\dots\dots (18)$$

c_3, c_4 are integration constants

And the surface velocity can be yields as.

$$\dot{h} = \frac{-\frac{1}{B} \exp(-Bt+c_3) + \frac{A}{B}}{h} \dots\dots\dots (19)$$

Characteristic of the Transient Motion with Friction

The second term in $\sqrt{\quad}$ of Equation (18) corresponds to Lucas-Washburn formula^[1] (this will be shown as L-W formula in the following) and Equation (18) approach L-W formula with $t \rightarrow \infty$.

Suppose $h = L$ and the velocity=0 at the time of $t=0$ like chapter 3, integration constants become $c_3=LnA$, $c_4=L^2 - 2A/B^2$. Figure 3 shows the graphs of Equation (18) (19) with the initial conditions of $L=3$ mm for instance and the surface velocity=0 at the time of $t=0$. Triangle marks

indicate height of the surface and circle marks indicate the surface velocity. Each parameters are same as ones in chapter 3 as below.

$$a=1e-3, b=2e-4, \rho=998, \sigma=7.28e-2, \theta=0$$

Though the surface velocity characteristic has steep gradient just after $t=0$ again, the peak number of it is as half as the above ideal number \sqrt{A} , and it decreases by the influence of the friction force. Regarding the difference between the surface velocity and \sqrt{A} as influence of the friction, it is clear that the motion is influenced by the friction very much.

Though L was tried to be changed from 3 mm to 0.01 mm like chapter 3 and then the peak number of the velocity increases with the decrease of L and it reaches about 0.65 which is approximately 70% of \sqrt{A} , the total tendency including being steep of the gradient just after $t=0$ is similar to one in chapter 3. Also the profile of Figure 3 shantly changes with the varying of a, b .

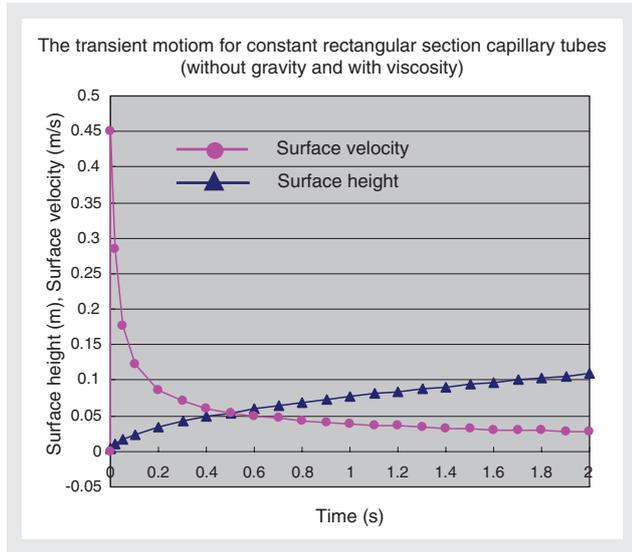


Figure 3 Surface transient actions with friction

In order to make the parameter tests to see the influence of a, b of rectangular section side lengths to the surface velocity, the surface velocity was studied as a estimation parameter of the time of $t=2$ for instance. $L=3$ mm is fixed and $b=0.2$ mm is also fixed on the study of influence of a , while $L=3$ mm is fixed and $a=1$ mm is fixed on the study of influence of b . Results can be seen in Figure 4, 5. Seeing them it is found that it is favorable to shorten the sampling time with the smaller a and bigger b . Remembering the first supposition contains big a and small b , the tendency above means the square configuration of the section would be the most favorable for a short time sampling. Generally the velocity distribution becomes parabolic one which is regarded as Hagen-Poiseuille flow in the flow between narrow gaps. So considering the narrower the gap is, the more steep the gradient near the wall of the parabolic velocity

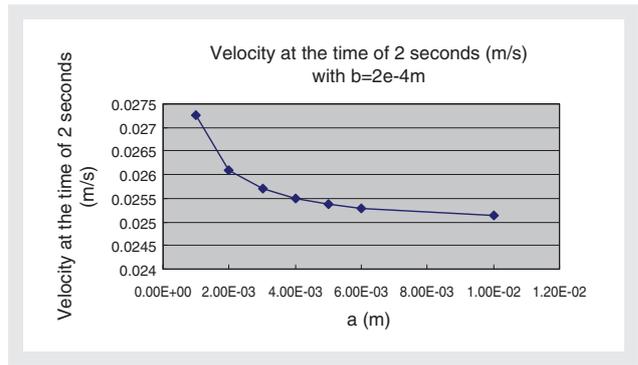


Figure 4 Parameter test result for a with b fixed

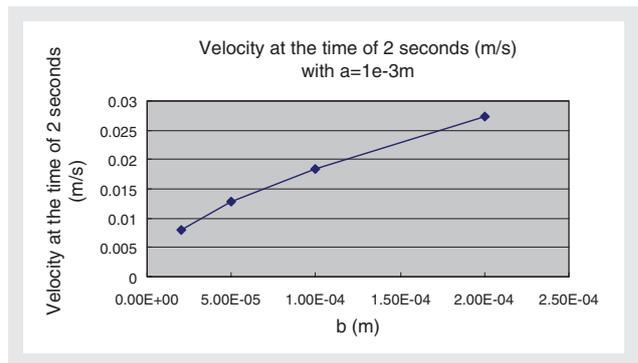


Figure 5 Parameter test result for b with a fixed

distribution becomes, above is because that the friction force can be smaller with elimination of the minimum gap part of the section as possible. Advancing this consideration, the best section shape would be circle. The similar calculations for circular tubes can be held as chapter 3 with results of B being as below with letting d be inner diameter of the tubes and A is same as one in Equation (11)

$$B = \frac{32\mu}{D^2\rho} \dots\dots\dots (20)$$

The similar transient motion corresponding to Figure 2, 3 can be got for the circular section tubes also. Then similar parameter tests for the inner diameter D was held. Figure 6 shows the results.

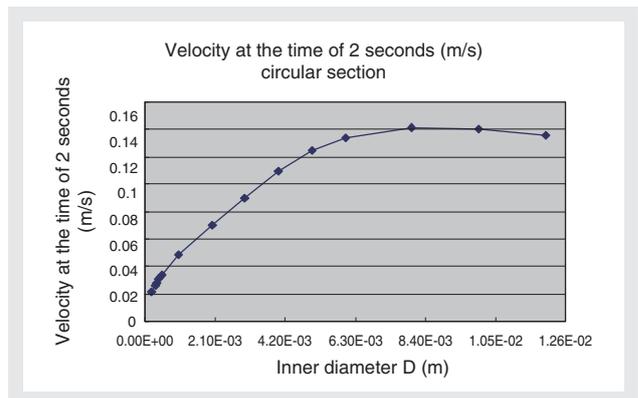


Figure 6 Parameter test result for D

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Study of the Transient Motion of Capillarity in Constant Section Tubes

The surface velocity of the time of $t=2$ was used as a estimation parameter as above The velocity has peak number around the inner diameter of approximate 8 mm. It would be because as followings. When the diameter is small, big influence of friction makes the velocity lows. Also when diameter becomes too big, it lets the weight to be sucked become big by the two power of the diameter, while the sucking force become only in proportional to the diameter, so the velocity becomes low also. Though it would be rather difficult in real world to use 8 mm inner diameter tube as a capillary tube used in horizontal way, which corresponding to neglecting gravity, it turns out that there exists the theoretically most proper diameter which lets the velocity in circular tubes be maximum. Figure 7 shows the corresponding graph to Figure 3 as a reference. Considering the ideal velocity of \sqrt{A} being about 0.17 and comparing with the Figure 2, friction influence is hardly observed which means there is possibility to decrease the friction influence very much theoretically.

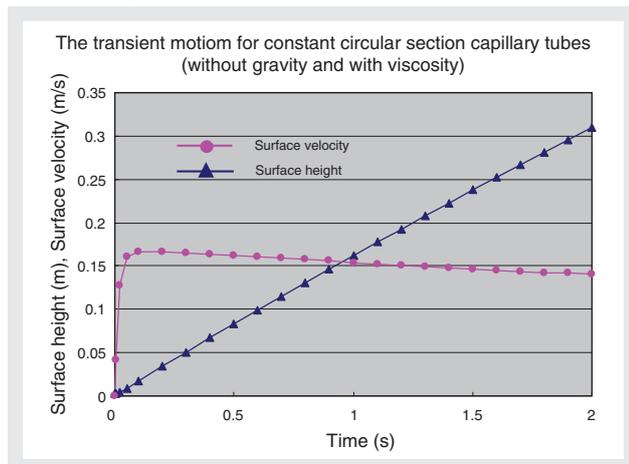


Figure 7 Surface transient actions with friction for circular section

Verification of Theoretical Results by CAE

The analytical solution which expresses the transient motion of capillarity got above was verified by CAE FVM calculation for circular tubes with constant section. The software used was STREAM Ver3.14 from Software CRADLE. and VOF method were used for the CAE free surface analysis. The parameters were used as the inner diameter of 0.4 mm and the initial height of $L=0.3$ mm and initial velocity of 0 m/s and $\rho=998$ kg/m³, $\sigma=7.28e-2$ N/m, and the contact angle of $\theta=10^\circ$. An expanded picture and velocity vectors around the surface at the

time of $t=0.01$ second is shown in Figure 8 for instance. The right-hand side picture of the center line is only shown by symmetrical configuration. The surface goes upwards, and the shape of the surface is shown by the contour line of VOF (the volume ratio of the liquid) being 0.5. The horizontal line in the picture shows the initial position of surface which is at the height of 3 mm.

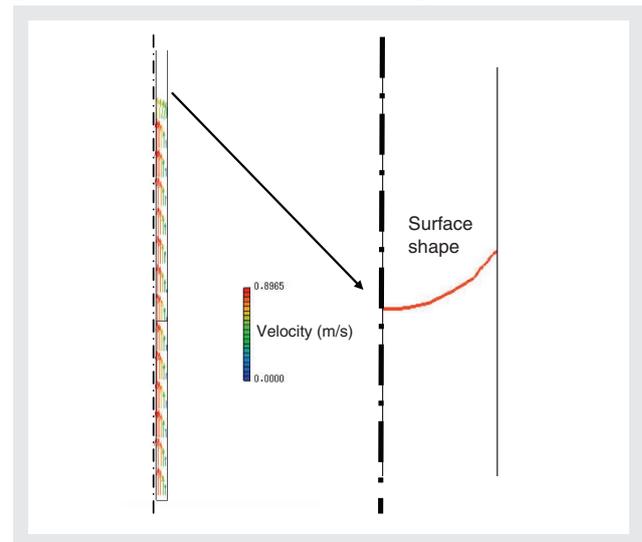


Figure 8 Velocity distribution and surface shape by capillarity

In Figure 9, 10 the transient motion of the surface height and surface velocity at the time from $t=0$ to 0.2 seconds are shown. The velocity results from CAE was estimated as the section average velocity at the section of which the height is 3 mm.

The transient motion of the surface height shows a good identity with one by CAE, and so does the transient motion of the surface velocity especially in the state in which the velocity rises from the initial velocity of 0 very steeply and afterward it declines slowly. It also means that the overshoot in CAE result just after $t=0$ is not a kind of error or unstableness in numerical calculation but the physical phenomenon through the theoretical analysis above. Though this overshoot has tendency to become

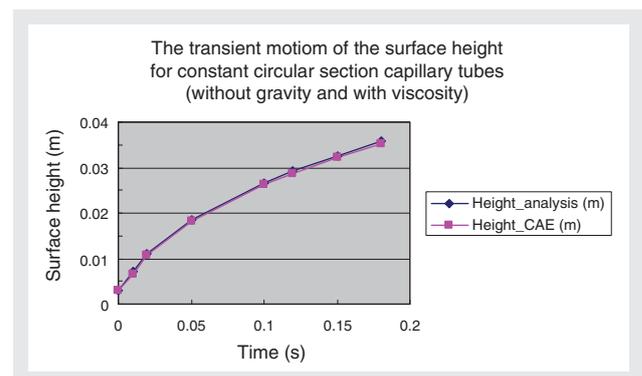


Figure 9 Comparison of the transient response of surface height

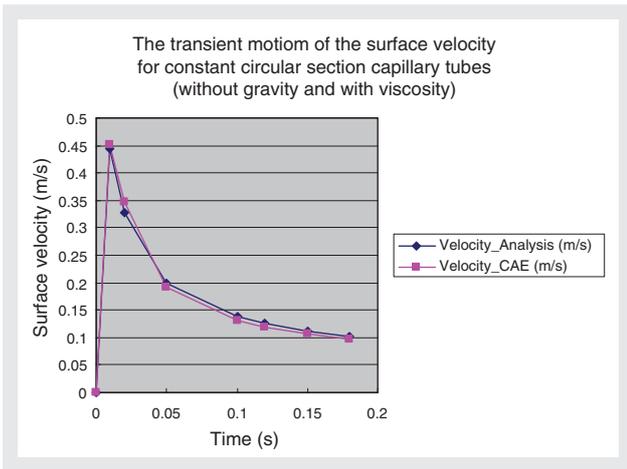


Figure 10 Comparison of the transient response of surface velocity

bigger in L-W formula than one in Figure 10 with bigger diameters or smaller viscosity, it coincides with one in L-W formula very much with smaller diameter than 1 mm or high viscosity by ten times even with the diameter of 3 mm. With often said information that L-W formula is more favorable with oil than water, these results seem to be consistent with fact that the second term in $\sqrt{\quad}$ of Equation (18) becomes bigger compared with the first term as the inner diameter and density become smaller and viscosity become bigger.

Summary of Conclusion

In capillarity the well-known heights at which the liquid surface stops by gravity exist as below,

$$h = \frac{4\sigma\cos\theta}{\rho g D} \text{ (circular tubes) } \dots\dots\dots (21)$$

$$h = \frac{2\sigma\cos\theta}{\rho g b} \text{ (between infinitely big planes) } \dots (22)$$

By this existence of the stopping height, sampling times has a certain limitations which make studys for sampling time and volume themselves hard, so in this paper we tried to make a theoretical approach without gravity which corresponds to the usage for capillary tubes in horizontal way of the transient motion of capillarity with constant rectangular or circular section. And we derived the formula which describes the transient motion of capillarity and also tried to show the relationship with Lucas-Washburn formula. As a consequence the followings resulted 1. There exists the maximum number \sqrt{A} (see Equation (5)) for the sampling velocity by capillarity in constant capillary tubes. 2. The square section capillary tube is favorable to shorten the sampling time by capillarity under the condition of the constant same rectangular section area for water for instance. 3.

There exists the theoretically most proper diameter for circular capillary tubes for water. The information above is hoped to be useful to designers for devices which utilize the capillary tubes.

Reference

[1] E. W. Washburn, Phys. Rev. 17, 273 (1921).



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